On Higher Order Mode Cutoff Frequencies in Elliptical Step Index Fibers

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Abstract —**Cutoff frequencies of several higher order modes in two-layer step index elliptical fiber are computed from a rigorous analysis of the boundary value problem. Cutoff characteristics of modes in rectangular and elliptical fibers in the literature are reviewed. Mode correspondence, field symmetries, and physical considerations are invoked to establish confidence in previously published results on the first higher order mode cut off in elliptical fibers. Very large birefringence fibers are shown to have low values for the first higher order mode cut off; hence, their usefulness in single-mode applications is limited.**

I. Introduction

THE POLARIZATION-preserving property of step index elliptical fibers was first demonstrated circa 1979 by Dyott *et al.* [1]. Since then the characteristics of optical fibers with elliptical cross section have been investigated by many researchers [2]–[9]. Rashleigh and Marrone [2] have evaluated the polarization-holding abilities of several fibers. In many applications, these fibers have been operated in the *single-mode* regime. The two dominant modes in an elliptical fiber, ${}_eHE_{11}$ and ${}_oHE_{11}$, have zero cutoff frequency. In a fiber with large birefringence, the propagation characteristics of these two modes differ substantially [1]; hence, the coupling between these two modes due to minor manufacturing imperfections, bends, etc., is not significant. The frequency of operation in a single-mode fiber should be maintained below the cutoff frequency of the first higher order mode. Thus, there is a need to accurately determine the first higher order mode cut off.

Cozens and Dyott [3] obtained this result from an approximate characteristic equation based on the existence of TM modes. Since elliptical fibers can support only hybrid modes, the applicability of their results is limited to quasi-circular fibers. Rengarajan and Lewis computed the first higher order mode cut off in two-layer and three-layer step index elliptical fibers [4], [5]. Their results were obtained from the exact characteristic equation in terms of Mathieu functions, modified Mathieu functions, and their derivatives. Later, Shevchenko analyzed the higher order mode cutoff frequencies of elliptical fibers using an approximate technique called shift formulas [6]. His results for the cutoff frequency of the first higher order mode, ${}_oEH_{01}$, were in close agreement with those given in [4]. Shevchenko's method is convenient for fibers of arbitrary cross section, whereas for elliptical fibers the Mathieu function method is numerically more efficient and accurate. Matsuhara determined the cutoff frequency of the first higher order mode in elliptical fibers by the boundary element analysis [7]. Matsuhara's results were in excellent agreement with those in [4]. A mode-matching method was employed by Miyamoto and Yasuura to investigate elliptically cored fibers [8]. Their results on the first higher order mode also were in excellent agreement with those in [4] and [7]. Recently Saad studied the elliptical fiber problem by employing an approximate point matching approach [9]. His results for the ${}_oEH_{01}$ mode cut off were in excellent agreement with similar data presented in [4] for the semiminor axis to semimajor axis ratio $b/a > 0.4$. Saad's results for $b/a < 0.4$ exhibit a substantial deviation from those of [4]. In this region, signified by large eccentricity, Saad's computed values of cutoff frequency for the ${}_oEH_{01}$ mode increase monotonically with decreasing b/a. Thus Saad's results indicate that it is possible to design a single-mode elliptical fiber with extremely large birefringence whereas physical considerations point otherwise. There remains a need to investigate the cutoff characteristics of higher order modes in elliptical fibers to study the single-mode regime in large-birefringence applications and to resolve the discrepancies between the results in [9] and those in [4]–[8].

In this paper, cutoff values of several higher order modes in the elliptical fiber are computed by solving the exact characteristic equation. From an investigation of mode correspondence and field symmetries in rectangular and elliptical fibers, and from physical considerations, confidence in previously published results on ${}_oEH_{01}$ mode cut off will be established. Results to be presented provide a complete understanding of the higher order mode cut off for very small values of b/a.

II. Higher Order Mode Cutoff

Fig. 1 shows the cross-sectional geometry of two-layer step index fibers investigated in this paper. The two-layer elliptical fiber shown in Fig. 1(a) consists of a core of refractive index n_1 and a cladding of refractive index n_2. The elliptical coordinate system used to solve this problem is shown in Fig. 2. A rigorous analysis of modes in this

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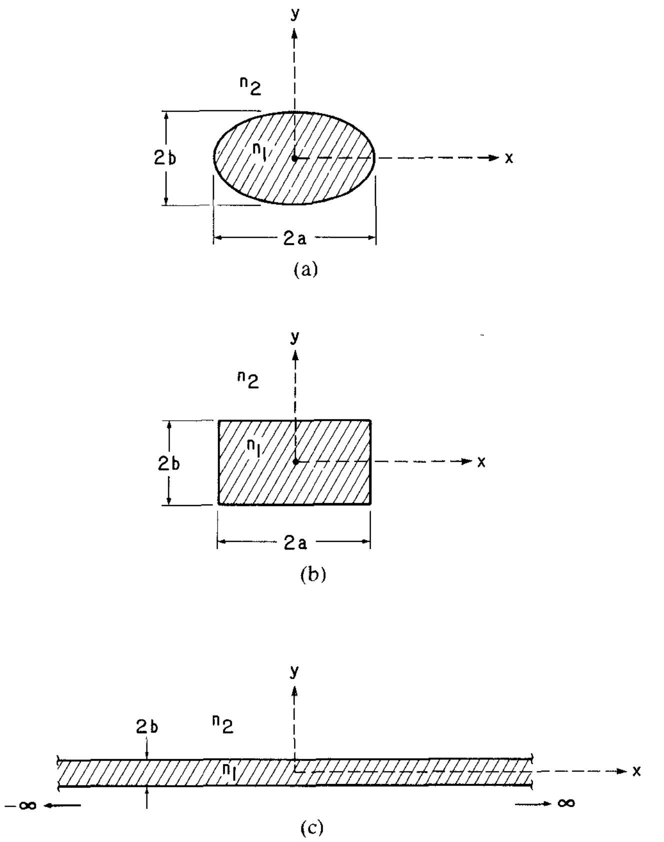

Fig. 1. Cross sections of step-index fibers. (a) Two-layer elliptical fiber. (b) Rectangular fiber. (c) Dielectric slab waveguide.

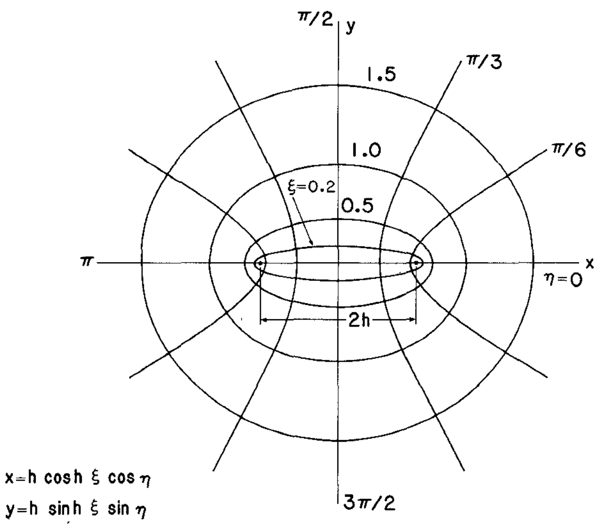

Fig. 2. The elliptical coordinate system.

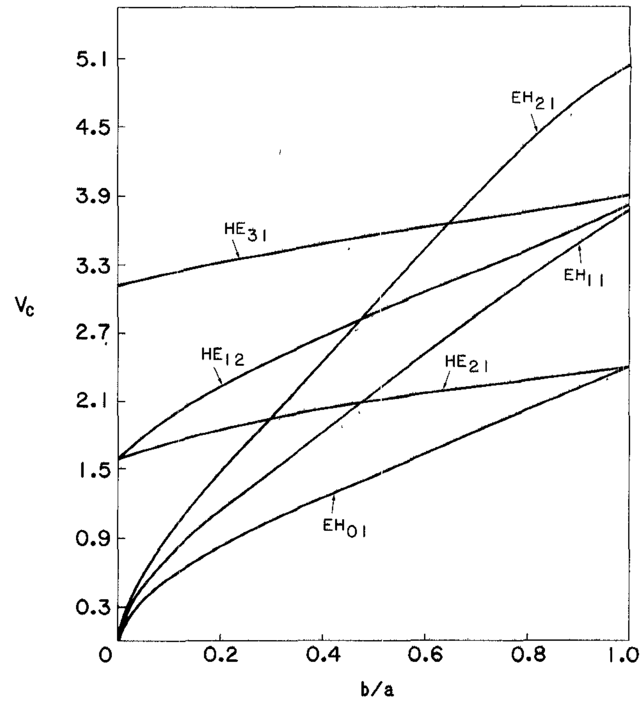

Fig. 3. Normalized cutoff values of six higher order odd modes in an elliptical fiber.

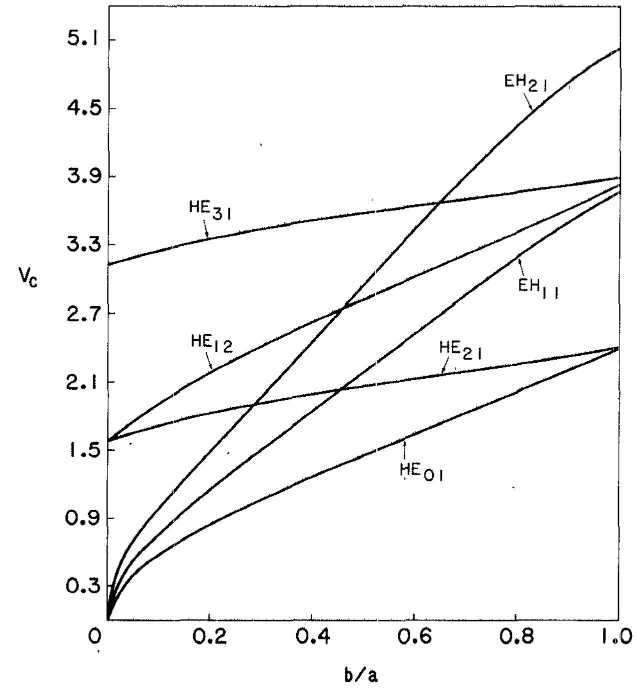

Fig. 4. Normalized cutoff values of six higher order even modes in an elliptical fiber.

waveguide results in a characteristic equation involving an infinite determinant in terms of Mathieu functions and modified Mathieu functions [10], [11]. The characteristic equation does not yield a simple closed-form expression under cutoff condition; hence, the cutoff characteristics are obtained numerically in this work. The cutoff frequency of a given mode is obtained by solving the characteristic equation $\beta \rightarrow n_2 k_0$, where β is the propagation constant of the mode under consideration and k_0 is the free-space wavenumber. Computed values of the normalized cutoff frequency,

$$V_c = 2\pi b\left[n_1^2 - n_2^2\right]^{1/2}/\lambda$$

are presented for 12 odd and even higher order modes as a function of axis ratio, b/a, in Figs. 3 and 4. Here λ is the free-space wavelength. Numerical values of $n_1 = 1.46$ and

$n_2 = 1.34$ have been assumed in these computations. The designation of these modes has been arrived at from a sequence of solutions for a quasi-circular fiber, $b/a \approx 1$ [11]. Since cutoff frequencies of many higher order modes were studied in this work, a revised and improved version of previously published algorithms [12] for computing the Mathieu functions and modified Mathieu functions was employed. In this new version, as many as 100 terms of sine and cosine series for Mathieu functions and Bessel function product series for modified Mathieu functions were considered. The characteristic determinantal equation has been previously shown to be rapidly convergent [4]. A determinant of order 5 has been shown to be adequate to compute the ${}_o\mathrm{EH}_{01}$ cut off to three figures even at high eccentricities. In this work determinants up to order 9 were computed.

The cutoff values presented in Figs. 3 and 4 are for the first 12 odd and even higher order modes at $b/a = 1$, where they correspond to those of a circular fiber. As b/a becomes smaller, the cutoff frequency of some other higher order modes drops within the range of values occupied by these 12 modes. For extremely small values of b/a, there are numerous other higher order modes having cutoff values between those of ${}_o\mathrm{EH}_{01}({}_e\mathrm{HE}_{01})$ and ${}_o\mathrm{HE}_{21}({}_e\mathrm{HE}_{21})$ exhibited in Figs. 3 and 4. As $b/a \to 0$, there is an infinite number of higher order modes exhibiting cutoff in the finite range discussed above. For all b/a values ${}_o\mathrm{EH}_{01}$ and ${}_e\mathrm{HE}_{01}$ are the first higher order odd and even modes respectively. The cutoff frequencies of four higher order modes given by Miyamoto and Yasuura [8] are in good agreement with similar results reported in this paper.

III. Mathieu Functions and Point-Matching Methods

The Mathieu function method, a natural choice for elliptical geometries, is an exact analytical approach in which the field expressions in the core and cladding regions are in terms of Mathieu and modified Mathieu functions and their derivatives [4], [5], [10], [11]. In this technique, field matching over the entire boundary between the core and the cladding is accomplished at a single value of the radial coordinate ξ, which is the argument of the modified Mathieu functions. The convergence behavior of the Mathieu functions and the characteristic equation have been previously discussed [12], [4]. The cutoff frequency of the ${}_o\mathrm{EH}_{01}$ mode presently computed with an improved version of Mathieu function programs (with a greater number of terms in the series and with a larger characteristic determinant) displayed in Fig. 3 is identical to that in [4]. There was no numerical problem experienced in computing V_c of the ${}_o\mathrm{EH}_{01}$ mode for b/a down to 0.05. It is believed that the results of the first higher order mode presented in [4] and in Fig. 3 of this paper are accurate to at least three figures.

The point-matching technique [9], [13] uses circular harmonic functions and Bessel functions for field expressions. In this method, field matching at the boundary between core and cladding is performed at a number of discrete

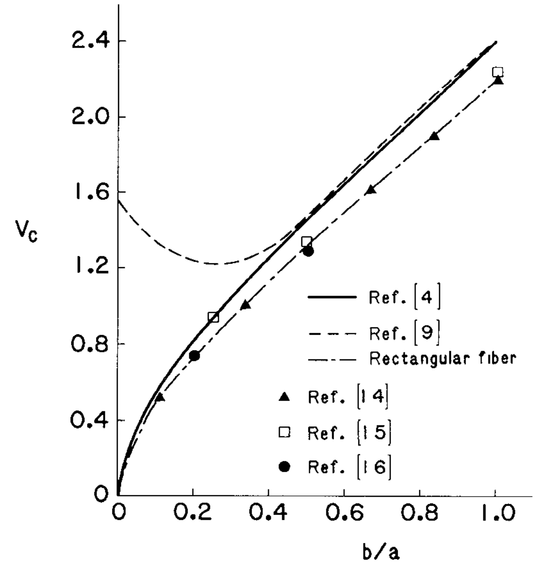

Fig. 5. First higher order mode cutoff parameter in elliptical and rectangular fibers.

points. Conceivably the point-matching approach requires large matrices and a great many match points and has convergence problems for small b/a values. Results of this method are questionable in this range. The cutoff values of the ${}_o\mathrm{EH}_{01}$ mode presented in [4] and [9] are in excellent agreement with other theoretical and experimental results [2], [6]–[8], [14] for $b/a > 0.40$. For $b/a < 0.40$, V_c of ${}_o\mathrm{EH}_{01}$ in [4] and in [9] are in substantial disagreement, as seen from Fig. 5. Unfortunately, there are no experimental data available for $b/a < 0.40$. The only other theoretical data available in the literature for $b/a < 0.40$ [6]–[8] are in close agreement with those in [4]. The cutoff frequency results of the ${}_o\mathrm{HE}_{21}$ mode presented in Saad's paper [9] are in good agreement with those in Fig. 3, whereas the results of ${}_o\mathrm{EH}_{11}$ in [9] show a substantial deviation from similar results given in Fig. 3 for small b/a values. It is interesting to note that the ${}_o\mathrm{EH}_{11}$ data in Fig. 3 are in good agreement with similar results reported in [8].

IV. Mode Correspondence and Limiting Behavior

In order to resolve the discrepancy in the cutoff results of the ${}_o\mathrm{EH}_{01}$ and ${}_o\mathrm{EH}_{11}$ modes between this paper and that of Saad, we consider mode correspondence in rectangular and elliptical fibers. Eyges *et al.* have found that the order in which the first dozen modes of an elliptical guide ($b/a = 0.5$) reach cutoff is the same as the order of modes in the rectangular guide ($b/a = 0.5$) whose cross section is shown in Fig. 1(b) [14]. Moreover, the actual values of the cutoff V have been found to be similar for the two shapes for equal values of b/a, n_1, and n_2. They have also found that the qualitative similarities between modes of the rectangular and elliptical guides approach numerical agreement as the axis ratio, b/a, becomes small. For $b/a = 0.1$, the first few modes have been found to be virtually indistinguishable [14]. In Fig. 5, the cutoff V numbers for the ${}_o\mathrm{EH}_{01}$ mode of an elliptical fiber [4], [9] and similar results for the first higher order mode in a rectangular fiber

TABLE I
FIELD SYMMETRIES AND CUTOFF V VALUES FOR DIELECTRIC SLAB WAVEGUIDES

Modes	Symmetry with respect to x-axis		V_c
	odd	even	
odd TM_n	E_z	H_x, E_y	0, π, ...
odd TE_n	H_z	H_y, E_x	0, π, ...
even TM_n	H_x, E_y	E_z	π/2, 3π/2...
even TE_n	H_y, E_x	H_z	π/2, 3π/2...

TABLE II
FIELD SYMMETRIES IN ELLIPTICAL FIBER WAVEGUIDES

Modes	symmetry with respect to major axis				V_c
	odd	even	odd	even	for the limiting case b/a→0.
odd hybrid modes	H_z, E_η, H_ξ	E_z, H_η, E_ξ	H_z, H_x, E_y	E_z, E_x, H_y	0, π . (E_x, H_y dominating) or π/2, 3π/2, . (H_x, E_y dominating)
even hybrid modes	E_z, H_η, E_ξ	H_z, E_η, H_ξ	E_z, E_x, H_y	H_z, H_x, E_y	0, π (H_x, E_y dominating) or π/2, 3π/2,. (E_x, H_y dominating)

[14]–[16] are plotted as a function of b/a. It is found that the first higher order mode cut off in the rectangular fiber is close to the ${}_oEH_{01}$ mode cut off in the elliptical fiber shown in Fig. 3 and in [4]. Saad's results for the ${}_oEH_{01}$ mode cut off, however, exhibit a substantial deviation from those of a rectangular fiber for small b/a values.

In a rectangular fiber, the first higher order mode, E_{21}^X or E_{21}^Y has a cutoff frequency that decreases with decreasing b/a [15]. As $b/a \rightarrow 0$, the cutoff frequencies of a doubly *infinite* number of modes, E_{21}^X, E_{21}^Y, E_{31}^X, E_{31}^Y, etc., approach zero. In the limiting case as $b/a = 0$, the rectangular fiber becomes a dielectric slab of infinite width and of thickness $2b$, shown in Fig. 1(c). From physical considerations, we should expect a one-to-one correspondence between modes in a dielectric slab guide and those in a rectangular fiber for the limiting case of $b/a = \epsilon$. Here ϵ is an infinitesimally small but *finite* positive quantity. In the former, modes have only one index, denoting the field variation in the y direction, i.e., ${}_oTE_n, {}_oTM_n, {}_eTE_n, {}_eTM_n$ [17]. The subscripts o and e denote odd and even symmetry respectively and n is the mode index. In the rectangular fiber ($b/a = \epsilon$) modes have two indices, denoting field variation in x and y directions, i.e., E_{mn}^X, E_{mn}^Y, where the index m denotes variation along the x direction. For $b/a = \epsilon$, corresponding to each mode in the dielectric slab guide there is an infinite number of modes in the rectangular fiber. As $\epsilon \rightarrow 0$, we find an infinite mode degeneracy corresponding to each mode in the dielectric slab guide. For example, E_{01}^X, E_{11}^X, E_{21}^X, etc., become E_1^X.

Based on the study of Eyges *et al.* [14] and from physical considerations, elliptical fibers are expected to have a similar limiting behavior when $b/a \rightarrow 0$. Fig. 5 shows that the first higher order mode cut off in a rectangular fiber and that in an elliptical fiber [4] are very nearly equal and both of them approach zero as $b/a \rightarrow 0$. The field symmetries for TE and TM modes in a dielectric slab waveguide are shown in Table I. Odd and even symmetries (with respect to the major axis) for hybrid modes in an elliptical fiber are given in Table II, where field components in elliptical coordinates and those in Cartesian coordinates near the center of the guide are shown. From considerations of mode correspondence and field symmetries, an infinite number of modes having zero cutoff are expected in the elliptical fiber for the limiting value of $b/a = 0$. Also, an infinite mode degeneracy is expected corresponding to *each* mode in the dielectric slab waveguide with a cutoff value of $\pi/2$, π, $3\pi/2$, etc. From Figs. 3 and 4 we find that six modes have $V_c = 0$, four modes have $V_c = \pi/2$, and two have $V_c = \pi$. This behavior is due to the fact that these are the first 12 higher order modes at $b/a = 1$. Obviously the *first 12* higher order modes near $b/a = 0$ would all have a $V_c \approx 0$.

Clearly Saad's results do not show any higher order mode exhibiting $V_c = 0$ at $b/a = 0$. His results on cutoff values of ${}_oEH_{01}$ and ${}_oEH_{11}$ modes for small b/a do not show the correct physical behavior and hence have serious limitations. These results might have been the cutoff values of some other higher order modes. Results presented in Figs. 3, 4, and 5 provide a complete physical understanding of the behavior of higher order modes cut off in elliptical fibers for small b/a values including the limiting case of $b/a = 0$. Also, it has been reaffirmed that fibers with very large birefringence have limited usefulness in single-mode applications, since they have a very small value for the first higher order mode cut off.

V. CONCLUSIONS

This paper has presented computed values of the cutoff frequencies of the first 12 odd and even higher order modes in a two-layer elliptical fiber. From an investigation of mode correspondence in rectangular and elliptical fibers, field symmetries, and physical considerations in the limiting case of $b/a = 0$, confidence in previously published results [4]–[8] is established. Cutoff values presented in a recent paper [9] are applicable for $b/a > 0.40$ whereas for smaller values of b/a those results are shown to have discrepancies. A very large birefringence elliptical fiber has a low value of the first higher order mode cut off and hence its utility in single mode application is limited.

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